Chapter 4.8: Antiderivatives

Definition

$F(x)$ is an *antiderivative* of $f(x)$ if $F'(x) = f(x)$.

Example: Find antiderivatives:

- $f(x) = 2x$

  $F(x) = x^2$

- $g(x) = \cos(x)$

  $G(x) = \sin(x)$

- $h(x) = 2e^{2x}$

  $H(x) = e^{2x}$

Antiderivatives are unique up to a constant

If $F'(x) = f(x)$ then $(F(x) + C)' = f(x)$, where C is a constant.

If we have some additional information about the antiderivative, we may be able to solve for C and get a unique antiderivative.

Notation

$$\underbrace{\frac{d}{dx}\left(f(x)\right)}_{\text{take derivative}} \qquad \underbrace{\int f(x)\,dx}_{\text{take (all) antiderivatives}}$$

The collection of all antiderivatives is known as *indefinite integral*.

$$\int x^5\,dx = \frac{1}{6}x^6 + C$$

$$\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x} + C$$

$$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$$

$$\int \frac{1}{x}\,dx = \ln(|x|) + C$$

$$\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$$

Initial value problem

Problem: Find $y(x)$ such that

$$\underbrace{\frac{dy}{dx} = f(x)}_{\text{differential equation}} \quad \text{and} \quad \underbrace{y(x_0) = y_0}_{\text{initial condition}}$$

Solution:

1. Compute *general solution*

$$F(x) + C$$

2. Find *particular solution* by solving

$$y_0 = F(x_0) + C$$

Example:

$$\frac{dy}{dx} = 10 - x, \quad y(0) = -1$$

$$y = \int \frac{dy}{dx}\, dx = \int 10 - x\, dx = 10x - \frac{1}{2}x^2 + C$$

Plugging in $-1 = y(0)$, and so

$$-1 = y(0) = C$$

Consequently,

$$y = 10x - \frac{1}{2}x^2 - 1$$

More entertaining $\int$

- $\displaystyle\int (e^x+1)^2\ dx \quad = \tfrac{1}{2}e^{2x} + 2e^x - x + C$

- $\displaystyle\int \frac{x^3-2x^2+x-3}{x^2}\ dx = \int x - 2 + \frac{1}{x} - \frac{3}{x^2}\ dx = \frac{1}{2}x^2 - 2x + 3x^{-1} + C$

- $\displaystyle\int \frac{2x}{1+x^4}\ dx = \arctan x^2 + C$

- $\displaystyle\int \tan(x)^2\ dx$
  $\displaystyle= \int \frac{\sin(x)^2}{\cos(x)^2}\ dx = \int \frac{1-\cos(x)^2}{\cos(x)^2}\ dx = \int \frac{1}{\cos(x)^2} - 1\ dx = \tan(x) - x + C$

- $\displaystyle\int \frac{e^{2x}-1}{e^x+1}\ dx \quad = \int \frac{(e^x-1)(e^x+1)}{e^x+1}\ dx = \int e^x - 1\ dx = e^x - x + C$